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INERTIA EFFECTS ON THE CRANK ANGULAR VELOCITY

OF A RECIPROCATING ENGINE

by

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PREFACE

This work is a continuation of studies in engine vibration analysis carried on during a course in Naval Engineering, Design, at the United States Naval Postgraduate School during the period January 1946 to June 1948.

The problem was suggested to the author by E. K. Gatcombe, of the Department of Mechanical Engineering, United States Naval Postgraduate School. This work is essentially designed for use in the solution of dynamic and torsional vibration problems. It is a refinement for expressing the velocity of the crank and has hitherto been considered arithmetically too cumbersome to use. The author believes this to be the first published solution of its kind, and feels that the work will pave the way to a more exact solution of turning effort due to inertia forces in an engine; and, hence, a more exact solution to the calculation of torsional vibration frequencies in the design of high speed reciprocating engines.

The author is indebted to E. K. Gatcombe, Ph.D., C. H. Denbow, Ph.D., W. R. Church, Ph.D. and C. C. Torrence, Ph.D. of the Postgraduate School Staff for advice and criticism in the solution of this problem. Particular gratitude is due Prof. Gatcombe for his painstaking check of the analysis in detail prior to editing.

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TABLE OF SYMBOLS AND ABBREVIATIONS

l Length of connecting rod.

r Length of crank arm.

p Ratio of crank arm length to length of connecting rod.

W Weight of crank arm.

W_f Weight of flywheel.

W_1 Equivalent weight of crank arm.

W_2 Weight of connecting rod.

W_3 Weight of piston and its parts.

W_y Equivalent weight of flywheel.

W_E Equivalent weight of parts point concentrated at the wrist pin.

W_c Equivalent weight of parts point concentrated at the crank pin.

m_c Equivalent mass at crank pin ($m_c = W_c/g$).

m_E Equivalent mass at wrist pin ($m_E = W_E/g$).

K Ratio of reciprocating mass to rotating mass ($K = m_E/m_c$).

g Acceleration due to gravity.

ω Angular velocity of crank arm at any time t.

ω_0 Angular velocity of crank arm at zero time.

S_E Displacement of wrist pin measured from top dead center downward.

V_E Linear velocity of W_E at any time.

V_{E_0} Linear velocity of W_E at zero time.

A_E Acceleration of W_E.

I_c Moment of inertia of W_c about its center of rotation.

U Kinetic energy of system.

θ Angle of crank with respect to vertical measured clockwise.

ϕ Angle of connecting rod with the vertical.

CHAPTER I

INTRODUCTION

The problem considered in this work concerns the nature of the crank velocity in a reciprocating engine as it is effected by the inertia of the masses of the system. The classical solution to the dynamics problem of an engine heretofore has considered the angular velocity of the crank or cranks to be constant. Such an approach has in the past given adequately fine results in the design of internal combustion and steam reciprocating machinery. Today with the advent of high speeds and accelerations, and the need for light weight propulsion units, the dynamic and vibration problems arising in an engine are far more serious. We can no longer tolerate the "safety factor" to cover our errors in assumptions. The urgent cry, particularily from our armed forces, for engines with greater life expectancy and guaranteed reliability must be answered; firstly, with more accurate preliminary analysis of a proposed design; and secondly, with better test methods of the finished product of design before the prototype is placed upon our production line. With the former consideration in mind, it is here endeavored to express analytically a solution to one small problem of a myriad of problems in better board design of engines. It is hoped that this solution will help pave the way to a more exact solution of the problem of torsional vibration in engine shaft and frame while the machine is in the design stage, and this, without an overwhelming encumberance of arithmetic computation.

CHAPTER II

THE SIMPLIFIED RECIPROCATING MECHANISM

The reciprocating mechanism with which we will deal is a simplified single cylinder engine with flywheel, crank, connecting rod, and piston with its associated parts. The following simplifications will apply in this analysis:

a. The system is frictionless.

b. The driving force will not be included.

c. The equivalent weight of the flywheel will be concentrated in the crank at the crank pin.

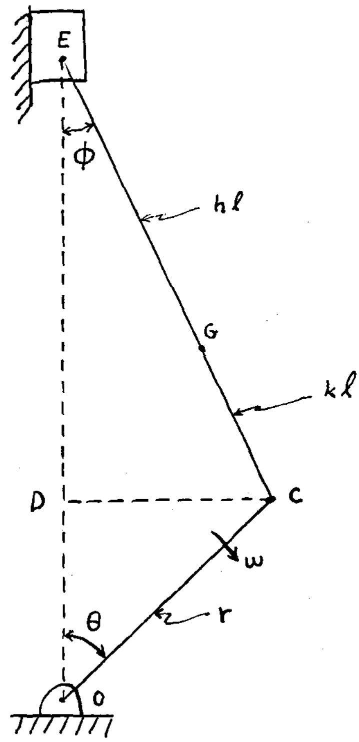

Fig. 1.

d. Static forces due to weight of parts are negligible in comparison to inertia forces.

e. An equivalent lumped mass system is presumed.

It is, of course, assumed from the start that angular velocity ω varies as some function of the crank angle θ and the masses involved in the system. To determine this function expressing ω is our fundamental problem.

CHAPTER III

THE ANGULAR VELOCITY OF THE CRANK

The kinematics of the system deals with the relative motion of three points (Fig. 1): the point O, which is the center of the crank shaft and crank pin circle; the point C, the center of the crank pin which moves in a circle; and the point E, the center of the wrist pin which is here taken to move along a vertical line as in an in line automobile engine. Let $EC = l$ be the length of the connecting rod and $OC = r$ be the crank radius.

The weight of the connecting rod W_2 can be concentrated at the points E and C by the equivalent weights kW_2 and hW_2 respectively where $W_2 = kW_2 + hW_2$. The ratios h and k depend upon the geometry of the connecting rod, the center of gravity G of the rod being at a distance hl from E and a distance kl from C. Having concentrated the weight hW_2 at C instead of at the center of percussion of the rod introduces into the system a residual couple acting on the connecting rod*. The weight of the piston and its parts will be taken as W_3 , hence the weight of the parts moving with the motion of E will be $W_E = W_3 + kW_2$. In like manner the weight of the parts rotating with the motion of C will be $W_c = W_1 + hW_2 + W_4$, where W_1 and W_4 are the equivalent weights of the crank W and flywheel W_f concentrated at C.

If the top dead center of the crank is taken as the origin of zero time, we may call the angular velocity of the crank in this position ω_o and the angle that the crank makes with the vertical equal to zero. The kinetic energy of the system may then be expressed by:

*See R. E. Root (1), Dynamics of Engine and Shaft, pp7.

$$U = 1/2\, I_c\, \omega^2 + 1/2\, m_E V_E^2 \qquad 1$$

Since the external forces are not considered in this solution, the energy in the system must be a constant, and:

$$1/2\, I_c\, \omega_0^2 + 1/2\, m_E V_{E_0}^2 = 1/2\, I_c\, \omega^2 + 1/2\, m_E V_E^2. \qquad 2$$

Since $V_{E_0} = 0$; $I_c = m_c r^2$; and, $V_E = \sin\theta\,(1 + p\cos\theta/\cos\phi)\, r\omega$, (see Appendix No. 1).

$$\omega_0^2/\omega^2 = 1 + (m_E/m_c)\sin^2\theta\,(1 + p\cos\theta/\cos\phi)^2. \qquad 3$$

Letting $m_E/m_c = K$;

$$\omega^2/\omega_0^2 = 1/[1 + K\sin^2\theta\,(1 + p\cos\theta/\cos\phi)^2]. \qquad 4$$

As will be apparent in a study of the dynamics of the moving mechanism, the forces and torques arising will always contain the angular velocity of the crank wherever it occurs as ω^2, therefore we will solve for ω^2 from equation (4) instead of ω in order to obtain a useful expression for the non-uniform angular velocity of the crank.

CHAPTER IV

THE ANGULAR VELOCITY OF THE CRANK EXPRESSED AS A FOURIER SERIES

In (4) let

$$\chi = K \sin^2\theta\,(1 + p\cos\theta/\cos\phi)^2, \qquad 5$$

so that expanding (4) by the binomial theorem gives us

$$\omega^2/\omega_0^2 = 1/(1+\chi) = 1 - \chi + \chi^2 - \chi^3 + \chi^4 \cdots. \qquad 6$$

But, we have from Fig. 1

$$CD = l\sin\phi = r\sin\theta. \qquad 7$$

And therefore

$$\cos\phi = (1 - \sin^2\phi)^{1/2} = (1 - p^2\sin^2\theta)^{1/2}, \qquad 8$$

the constant p being the ratio of the crank radius to the length of the connecting rod $pl = r$. Expanding $1/\cos\phi$ by the binomial theorem we have

$$1/\cos\phi = (1 - p^2\sin^2\theta)^{-1/2} = (1 + 1/2\,p^2\sin^2\theta + 3/8\,p^4\sin^4\theta + 5/16\,p^6\sin^6\theta + 35/128\,p^8\sin^8\theta + 63/256\,p^{10}\sin^{10}\theta \cdots), \qquad 9$$

which we may rewrite as

$$1/\cos\phi = (1 + a_2 p^2\sin^2\theta + a_4 p^4\sin^4\theta + a_6 p^6\sin^6\theta + a_8 p^8\sin^8\theta + a_{10} p^{10}\sin^{10}\theta \cdots). \qquad 10$$

From the general identity for even powered sines

$$\sin^n\theta = \left[(-1)^{n/2}/2^{n-1}\right]\left[\cos n\theta - \binom{n}{1}\cos(n-2)\theta + \binom{n}{2}\cos(n-4)\theta - \binom{n}{3}\cos(n-6)\theta + \cdots + \left((-1)^{n/2}/2\right)\binom{n}{n/2}\right],$$

we can expand each term of (10) into a cosine series as follows:

$$\begin{aligned}
a_{10}p^{10}\sin^{10}\theta &= a_{10}\frac{p^{10}}{2^9}\left[-\cos 10\theta + \binom{10}{1}\cos 8\theta - \binom{10}{2}\cos 6\theta + \binom{10}{3}\cos 4\theta - \binom{10}{4}\cos 2\theta + \frac{1}{2}\binom{10}{5}\right]\\
a_8p^8\sin^8\theta &= a_8\frac{p^8}{2^7}\left[\cos 8\theta - \binom{8}{1}\cos 6\theta + \binom{8}{2}\cos 4\theta - \binom{8}{3}\cos 2\theta + \frac{1}{2}\binom{8}{4}\right]\\
a_6p^6\sin^6\theta &= a_6\frac{p^6}{2^5}\left[-\cos 6\theta + \binom{6}{1}\cos 4\theta - \binom{6}{2}\cos 2\theta + \frac{1}{2}\binom{6}{3}\right]\\
a_4p^4\sin^4\theta &= a_4\frac{p^4}{2^3}\left[\cos 4\theta - \binom{4}{1}\cos 2\theta + \frac{1}{2}\binom{4}{2}\right]\\
a_2p^2\sin^2\theta + 1 &= 1 + a_2\frac{p^2}{2}\left[-\cos 2\theta + \frac{1}{2}\binom{2}{1}\right].
\end{aligned} \qquad 11$$

If we omit powers of p above the 10th we then have

$$1/\cos\phi = A_0 + A_2\cos 2\theta + A_4\cos 4\theta + A_6\cos 6\theta \cdots, \qquad 12$$

where by collecting terms from (11) the A's are given by the power series in p,

$$\begin{aligned}
A_{10} &= -\left[\cdots a_{10}\frac{p^{10}}{2^9}\right]\\
A_8 &= \left[\cdots \binom{10}{1}a_{10}\frac{p^{10}}{2^9} + a_8\frac{p^8}{2^7}\right]\\
A_6 &= -\left[\cdots \binom{10}{2}a_{10}\frac{p^{10}}{2^9} + \binom{8}{1}a_8\frac{p^8}{2^7} + a_6\frac{p^6}{2^5}\right]\\
A_4 &= \left[\cdots \binom{10}{3}a_{10}\frac{p^{10}}{2^9} + \binom{8}{2}a_8\frac{p^8}{2^7} + \binom{6}{1}a_6\frac{p^6}{2^5} + a_4\frac{p^4}{2^3}\right]\\
A_2 &= -\left[\cdots \binom{10}{4}a_{10}\frac{p^{10}}{2^9} + \binom{8}{3}a_8\frac{p^8}{2^7} + \binom{6}{2}a_6\frac{p^6}{2^5} + \binom{4}{1}a_4\frac{p^4}{2^3} + a_2\frac{p^2}{2}\right]\\
A_0 &= \left[\cdots \binom{10}{5}a_{10}\frac{p^{10}}{2^9} + \binom{8}{4}a_8\frac{p^8}{2^7} + \binom{6}{3}a_6\frac{p^6}{2^5} + \binom{4}{2}a_4\frac{p^4}{2^3} + \binom{2}{1}a_2\frac{p^2}{2} + 1\right].
\end{aligned} \qquad 13$$

At first glance the set of series in (13) seems rather formidable to the prospective computer who is about to find a solution for ω^2. In an actual engine however, the powers of p above the 6th would seldom occur and then only in A_0 and A_2. The more detailed set is here shown only for the purpose of outlining the general pattern of these coefficients.

With $1/\cos\phi$ now expanded into a cosine series we can proceed with the expansion of x. Substituting (12) for $1/\cos\phi$ we have

$$\rho\cos\theta/\cos\phi = \rho\cos\theta\left[A_0 + A_2\cos 2\theta + A_4\cos 4\theta + \cdots\right],$$

and since

$$\cos m\theta \cos n\theta = \frac{1}{2}\left[\cos(m+n)\theta + \cos(m-n)\theta\right], \qquad 14$$

we have

$$\rho\cos\theta/\cos\phi = \frac{p}{2}\left[(2A_0 + A_2)\cos\theta + (A_2 + A_4)\cos 3\theta + (A_4 + A_6)\cos 5\theta \cdots\right],$$

and

$$(1 + p\cos\theta/\cos\phi) = 1 + B_1\cos\theta + B_3\cos 3\theta + B_5\cos 5\theta \cdots, \qquad 15$$

wherein, if n = 1, 2, 3, 4, ---

$$B_1 = p/2\left[2A_0 + A_2\right]; \qquad B_{2n+1} = \frac{p}{2}\left[A_{2n} + A_{2n+2}\right]. \qquad 16$$

Taking the product of (15) with $\sin\theta$ by the identity

$$\sin m\theta \cos n\theta = \frac{1}{2}\left[\sin(m+n)\theta + \sin(m-n)\theta\right],$$

results in

$$\sin\theta\,(1+\rho\cos\theta/\cos\phi) = \sin\theta\,(1 + B_1\cos\theta + B_3\cos 3\theta + B_5\cos 5\theta \cdots],$$

$$\sin\theta\,(1+\rho\cos\theta/\cos\phi) = \sin\theta + \frac{B_1}{2}\sin 2\theta + \frac{B_3}{2}\sin 4\theta + \frac{B_3}{2}\sin(-2\theta),$$

$$+ \frac{B_5}{2}\sin 6\theta + \frac{B_5}{2}\sin(-4\theta) + \frac{B_7}{2}\sin 8\theta + \frac{B_7}{2}\sin(-6\theta)\cdots,$$

$$\sin\theta(1+\rho\cos\theta/\cos\phi) = \sin\theta + \left(\frac{B_1}{2} - \frac{B_3}{2}\right)\sin 2\theta + \left(\frac{B_3}{2} - \frac{B_5}{2}\right)\sin 4\theta$$

$$+\left(\frac{B_5}{2} - \frac{B_7}{2}\right)\sin 6\theta + \left(\frac{B_7}{2} - \frac{B_9}{2}\right)\sin 8\theta \cdots,$$

$$\sin\theta(1+\rho\cos\theta/\cos\phi) = C_1\sin\theta + C_2\sin 2\theta + C_4\sin 4\theta + C_6\sin 6\theta \cdots. \qquad 17$$

Then, substituting (17) into (5)

$$\chi = K\sin^2\theta(1+\rho\cos\theta/\cos\phi)^2 = K\left[C_1\sin\theta + C_2\sin 2\theta + C_4\sin 4\theta + C_6\sin 6\theta\cdots\right]^2$$

$$\chi = K\,(C_1^2\sin^2\theta + C_2^2\sin^2 2\theta + C_4^2\sin^2 4\theta + C_6^2\sin^2 6\theta \cdots) +$$

$$KC_1\sin\theta\,(C_2\sin 2\theta + C_4\sin 4\theta + C_6\sin 6\theta + C_8\sin 8\theta\cdots) +$$

$$KC_2\sin 2\theta(C_1\sin\theta + C_4\sin 4\theta + C_6\sin 6\theta + C_8\sin 8\theta\cdots) +$$

$$KC_4\sin 4\theta(C_1\sin\theta + C_2\sin 2\theta + C_6\sin 6\theta + C_8\sin 8\theta\cdots) +$$

$$KC_6\sin 6\theta(C_1\sin\theta + C_2\sin 2\theta + C_4\sin 4\theta + C_8\sin 8\theta\cdots) + \cdots. \qquad 18$$

Then by the identity

$$\sin m\theta\,\sin n\theta = 1/2\,(\cos(m-n)\theta - \cos(m+n)\theta),$$

(18) becomes

$$\chi = \frac{K}{2}\,(C_1^2 - C_1^2\cos 2\theta + C_2^2 - C_2^2\cos 4\theta + C_4^2 - C_4^2\cos 8\theta + C_6^2 - C_6^2\cos 12\theta\cdots) +$$

$$\frac{K}{2}(C_1C_2\cos\theta - C_1C_2\cos 3\theta + C_1C_4\cos 3\theta - C_1C_4\cos 5\theta + C_1C_6\cos 5\theta - C_1C_6\cos 7\theta\cdots) +$$

$$\frac{K}{2}\left(c_2c_1\cos\theta - c_2c_1\cos 3\theta + c_2c_4\cos 2\theta - c_2c_4\cos 6\theta + c_2c_6\cos 4\theta - c_2c_6\cos 8\theta + \cdots\right) +$$
$$\frac{K}{2}\left(c_4c_1\cos 3\theta - c_4c_1\cos 5\theta + c_4c_2\cos 2\theta - c_4c_2\cos 6\theta + c_4c_6\cos 2\theta - c_4c_6\cos 10\theta + \cdots\right) +$$
$$\frac{K}{2}\left(c_6c_1\cos 5\theta - c_6c_1\cos 7\theta + c_6c_2\cos 4\theta - c_6c_2\cos 8\theta + c_6c_4\cos 2\theta - c_6c_4\cos 10\theta + \cdots\right) + \cdots.$$

Then

$$\begin{aligned}
x = {} & \frac{K}{2}\left(c_1^2 + c_2^2 + c_4^2 + c_6^2 \cdots\right) + K\left(c_1c_2\right)\cos\theta + \\
& \frac{K}{2}\left(-c_1^2 + 2c_2c_4 + 2c_4c_6 + 2c_6c_8 + \cdots\right)\cos 2\theta + K\left(-c_1c_2 + c_1c_4\right)\cos 3\theta + \\
& \frac{K}{2}\left(-c_2^2 + 2c_2c_6 + 2c_4c_8 + 2c_6c_{10} + \cdots\right)\cos 4\theta + K\left(-c_1c_4 + c_1c_6\right)\cos 5\theta + \\
& \frac{K}{2}\left(-2c_2c_4 + 2c_2c_8 + 2c_4c_{10} + 2c_6c_{12} + \cdots\right)\cos 6\theta + K\left(-c_1c_6 + c_1c_8\right)\cos 7\theta + \\
& \frac{K}{2}\left(-2c_2c_6 - c_4^2 + 2c_2c_{10} + 2c_4c_{12} + \cdots\right)\cos 8\theta + K\left(-c_1c_8 + c_1c_{10}\right)\cos 9\theta + \\
& \frac{K}{2}\left(-2c_2c_8 - 2c_4c_6 + 2c_2c_{12} + 2c_4c_{14} + \cdots\right)\cos 10\theta + \cdots,
\end{aligned} \qquad 19$$

or

$$x = D_0 + D_1\cos\theta + D_2\cos 2\theta + D_3\cos 3\theta + D_4\cos 4\theta + \cdots. \qquad 20$$

For an engine with values $p = .25$ and $K = .12$ which can be considered representative, the value of D_6 will be less than 10^{-4} and will permit calculation of x accurately to six decimal places.

Equation (6) gives us

$$\omega^2/\omega_0^2 = 1 - x + x^2 - x^3 + x^4 + \cdots$$

which from (20) becomes

$$\omega^2/\omega_0^2 = 1 - \left(D_0 + D_1\cos\theta + D_2\cos 2\theta + \cdots\right) + \left(D_0 + D_1\cos\theta + D_2\cos 2\theta + \cdots\right)^2 + \cdots \qquad 21$$

Expanding

$$\left(D_0 + D_1 \cos\theta + D_2 \cos 2\theta + D_3 \cos 3\theta + \cdots\right)^2,$$

we get

$$\begin{aligned}
&(D_0^2 + D_0 D_1 \cos\theta + D_0 D_2 \cos 2\theta + D_0 D_3 \cos 3\theta + \cdots) + \\
&D_1 \cos\theta\,(D_0 + D_1 \cos\theta + D_2 \cos 2\theta + D_3 \cos 3\theta + \cdots) + \\
&D_2 \cos 2\theta\,(D_0 + D_1 \cos\theta + D_2 \cos 2\theta + D_3 \cos 3\theta + \cdots) + \\
&D_3 \cos 3\theta\,(D_0 + D_1 \cos\theta + D_2 \cos 2\theta + D_3 \cos 3\theta + \cdots) + \cdots,
\end{aligned}$$

and by equation (14)

$$\begin{aligned}
&(D_0^2 + D_1^2/2 + D_2^2/2 + D_3^2/2 + \cdots) + \\
&(2D_0 D_1 + D_1 D_2 + D_2 D_3 + D_3 D_4 + \cdots)\cos\theta + \\
&(2D_0 D_2 + D_1^2/2 + D_1 D_3 + D_2 D_4 + \cdots)\cos 2\theta + \\
&(2D_0 D_3 + D_1 D_2 + D_1 D_4 + D_2 D_5 + \cdots)\cos 3\theta + \\
&(2D_0 D_4 + D_1 D_3 + D_2^2/2 + D_1 D_5 + D_2 D_6 + \cdots)\cos 4\theta + \\
&(2D_0 D_5 + D_1 D_4 + D_2 D_3 + D_1 D_6 + D_2 D_7 + \cdots)\cos 5\theta + \\
&(2D_0 D_6 + D_1 D_5 + D_2 D_4 + D_3^2/2 + D_1 D_7 + \cdots)\cos 6\theta + \cdots
\end{aligned} \qquad 22$$

Substituting (22) into (21) gives

$$\begin{aligned}
\omega^2/\omega_0^2 = {} &(1 - D_0 + D_0^2 + D_1^2/2 + D_2^2/2 + D_3^2/2 + \cdots) + \\
&(-D_1 + 2D_0 D_1 + D_1 D_2 + D_2 D_3 + D_3 D_4 + \cdots)\cos\theta + \\
&(-D_2 + 2D_0 D_2 + D_1^2/2 + D_1 D_3 + D_2 D_4 + \cdots)\cos 2\theta + \\
&(-D_3 + 2D_0 D_3 + D_1 D_2 + D_1 D_4 + D_2 D_5 + \cdots)\cos 3\theta +
\end{aligned}$$

$$\left(-D_4 + 2D_0D_4 + D_1D_3 + D_2^2/2 + D_1D_5 + D_2D_6 + \cdots\right)\cos 4\theta +$$

$$\left(-D_5 + 2D_0D_5 + D_1D_4 + D_2D_3 + D_1D_6 + D_2D_7 + \cdots\right)\cos 5\theta +$$

$$\left(-D_6 + 2D_0D_6 + D_1D_5 + D_2D_4 + D_3^2/2 + D_1D_7 + \cdots\right)\cos 6\theta + \cdots. \qquad 23$$

which can be rewritten as

$$\omega^2/\omega_0^2 = F_0 + F_1\cos\theta + F_2\cos 2\theta + F_3\cos 3\theta + \cdots. \qquad 24$$

where the F's can be expressed as a power series in p by use of substitutions from (23), (19), (17), (16), (13) and (9), (see Chapter V).

CHAPTER V

THE FOURIER COEFFICIENTS

FOR THE ANGULAR VELOCITY OF A CRANK

In determining the coefficients of the Fourier series which represents the angular velocity of a crank as $\omega^2/\omega_0^2 = f\ (\theta)$, repeated products and sums of trigonometric series must be taken to arrive at a usable expression. To express the complete result of such a combination for even one coefficient of ω^2/ω_0^2 would be overwhelming. It therefore behooves the analyzer to know the magnitude of all of the constants in each of the series that enters the final combination. For this purpose representative values p = .25 and K = .12 are taken in this analysis, and each coefficient is determined with sufficient number of terms to give accuracy to five and in some instances six decimal places. This degree of accuracy has been taken to give a more general result and to allow for increase in the magnitudes of K and p above that which is considered representative by the author. The complete solution for F_0 follows.

From (23)

$$F_0 = 1 - D_0 + D_0^2 + D_1^2/2 + D_2^2/2 + D_3^2/2 + \cdots \qquad 25$$

Substituting for D's from (19) and for C's from (17) results in

$$D_0 = \frac{K}{2}\left(C_1^2 + C_2^2 + C_4^2 + \cdots\right) = \frac{K}{8}\left(4 + B_1^2 - 2B_1B_3 + B_3^2 - 2B_3B_5 + \cdots\right)$$

$$D_1 = K\left(C_1C_2\right) = \frac{K}{2}\left(B_1 - B_3\right)$$

$$D_2 = \frac{K}{2}\left(-C_1^2 + 2C_2C_4 + 2C_4C_6 + 2C_6C_8 + \cdots\right) = \frac{K}{8}\left(-4 + 2B_1B_3 - 2B_3^2 - 2B_1B_5 + 4B_3B_5 + \cdots\right)$$

$$D_3 = K\left(-C_1C_2 + C_1C_4\right) = \frac{K}{2}\left(-B_1 + 2B_3 - B_5\right)$$

$$D_4 = \frac{K}{2}(-C_2^2 + 2C_2C_6 + 2C_4C_8 + 2C_6C_{10} + \cdots) = \frac{K}{8}(-B_1^2 + 2B_1B_3 - B_3^2 + 2B_1B_5 - 2B_3B_5 + \cdots)$$

$$D_5 = K(-C_1C_4 + C_1C_6) = \frac{K}{2}(-B_3 + 2B_5 - B_7)$$

$$D_6 = \frac{K}{2}(-C_2C_4 + 2C_2C_8 + 2C_4C_{10} + 2C_6C_{12}) = \frac{K}{4}(B_1B_5 - B_1B_3 + B_3^2 - B_3B_5 + \cdots) \qquad 26$$

Substituting (26) into (25), simplifying and suppressing terms of magnitude less than 10^{-6} gives the following steps.

$$F_0 = 1 - \frac{K}{8}(4 + B_1^2 - 2B_1B_3 + 2B_3^2 + \cdots) + \frac{K^2}{64}(4 + B_1^2 - 2B_1B_3 + 2B_3^2 + \cdots)^2$$
$$+ \frac{K^2}{8}(B_1 - B_3)^2 + \frac{K^2}{128}(-4 + 2B_1B_3 - 2B_3^2 + \cdots)^2 + \frac{K^2}{8}(-B_1 + 2B_3 - B_5)^2 + \cdots$$

$$F_0 = 1 - \frac{K}{8}(4 + B_1^2 - 2B_1B_3 + 2B_3^2 + \cdots) + \frac{K^2}{128}(32 + 16B_1^2 - 32B_1B_3 + 32B_3^2 + \cdots)$$
$$+ \frac{K^2}{128}(16B_1^2 - 32B_1B_3 + 16B_3^2) + \frac{K^2}{128}(16 - 16B_1B_3 + 16B_3^2 + \cdots)$$
$$+ \frac{K^2}{128}(16B_1^2 - 64B_1B_3 + 64B_3^2 + \cdots)$$

$$F_0 = 1 - \frac{K}{8}(4 + B_1^2 - 2B_1B_3 + 2B_3^2 + \cdots) + \frac{K^2}{8}(3 + 3B_1^2 - 9B_1B_3 + 8B_3^2 + \cdots)$$

$$F_0 = 1 - \frac{K}{2} + \frac{3}{8}K^2 + \left(\frac{3K^2 - K}{8}\right)B_1^2 - \left(\frac{9K^2 - 2K}{8}\right)B_1B_3 + \left(\frac{4K^2 - K}{4}\right)B_3^2 + \cdots \qquad 27$$

In (27) substitute from (16) for the B's and assume the following identities

$$\frac{3K^2 - K}{8} = \alpha \quad ; \quad -\frac{9K^2 - 2K}{8} = \beta \quad ; \quad \frac{4K^2 - K}{4} = \Upsilon,$$

so that by again suppressing terms of lesser magnitude,

$$F_0 = 1 - \frac{K}{2} + \frac{3K^2}{8} + \alpha\frac{\rho^2}{4}(4A_0^2 + 4A_0A_2 + A_2^2)$$
$$+ \beta\frac{\rho^2}{4}(2A_0A_2 + A_2^2 + 2A_0A_4 + A_2A_4) + \Upsilon\frac{\rho^2}{4}(A_2^2 + 2A_2A_4 + A_4^2) + \cdots$$

$$F_0 = 1 - \frac{K}{2} + \frac{3K^2}{8} + \alpha\rho^2A_0^2 + \left(\alpha + \frac{\beta}{2}\right)\rho^2A_0A_2 + (\alpha + \beta + \Upsilon)\frac{\rho^2}{4}A_2^2 + \frac{\beta}{2}\rho^2A_0A_4 + \cdots \qquad 28$$

From (13)

$$A_0 = \left[\cdots \binom{4}{2} \frac{a_4}{2} \frac{p^4}{2^3} + \binom{2}{1} \frac{a_2}{2} \frac{p^2}{2} + 1 \right] = \left[\cdots 6 a_4 \left(\frac{p}{2}\right)^4 + 2 a_2 \left(\frac{p}{2}\right)^2 + 1 \right]$$

$$A_2 = \left[\cdots - \binom{4}{1} a_4 \frac{p^4}{2^3} - a_2 \frac{p^2}{2} \right] = \left[\cdots - 8 a_4 \left(\frac{p}{2}\right)^4 - 2 a_2 \left(\frac{p}{2}\right)^2 \right]$$

$$A_4 = \left[\cdots a_4 \frac{p^4}{2^3} \right] = \left[\cdots 2 a_4 \left(\frac{p}{2}\right)^4 \right],$$

so that by substitution, (28) becomes

$$\begin{aligned} F_0 = {} & 1 - \frac{K}{2} + \frac{3K^2}{8} + \alpha p^2 \left[1 + 4 a_2 \left(\frac{p}{2}\right)^2 + (12 a_4 + 4 a_2^2) \left(\frac{p}{2}\right)^4 + \cdots \right] \\ & + \left(\alpha + \frac{\beta}{2}\right) p^2 \left[- 2 a_2 \left(\frac{p}{2}\right)^2 - (8 a_4 + 4 a_2^2) \left(\frac{p}{2}\right)^4 + \cdots \right] \\ & + (\alpha + \beta + \gamma) p^2 \left[a_2^2 \left(\frac{p}{2}\right)^4 + \cdots \right] + \beta p^2 \left[a_4 \left(\frac{p}{2}\right)^4 \right] \end{aligned}$$

$$\begin{aligned} F_0 = {} & 1 - \frac{K}{2} + \frac{3K^2}{8} + \alpha p^2 + \left[4 a_2 \alpha - 2 a_2 \left(\alpha + \frac{\beta}{2}\right) \right] \frac{p^4}{4} \\ & + \left[(12 a_4 + 4 a_2^2)\alpha - (8 a_4 + 4 a_2^2)\left(\alpha + \frac{\beta}{2}\right) + a_2^2 (\alpha + \beta + \gamma) + a_4 \beta \right] \frac{p^6}{16} \\ & + \cdots \end{aligned}$$

$$\begin{aligned} F_0 = {} & 1 - \frac{K}{2} + \frac{3K^2}{8} + \alpha p^2 + \left[(2\alpha - \beta) a_2 \right] \frac{p^4}{4} \\ & + \left[(4\alpha - 3\beta) a_4 + (\alpha - \beta + \gamma) a_2^2 \right] \frac{p^6}{16} + \cdots \end{aligned} \qquad 29$$

Substituting into (29) the identities for α, β, and γ and from (9) the values for the a's gives

$$\begin{aligned} F_0 = {} & 1 - \frac{K}{2} + \frac{3K^2}{8} + \frac{3K^2 - K}{8} p^2 + (15K^2 - 4K) \frac{p^4}{64} \\ & + \left(\frac{157K^2 - 40K}{16} \right) \left(\frac{p}{2}\right)^6 + \cdots \end{aligned} \qquad 30$$

It is to be noted that the first four terms of (30) give three decimal place accuracy for a considerable increase in the magnitude of p and K over that suggested earlier in this analysis.

The coefficients for the remainder of the series can be determined in a similar manner to that shown for F_0. Those coefficients which are of significant magnitude are herewith quoted for reference.

$$
\begin{aligned}
F_0 &= 1 - \frac{K}{2} + 3\frac{K^2}{8} + \frac{3K^2-K}{2}\left(\frac{p}{2}\right)^2 + \frac{15K^2-4K}{4}\left(\frac{p}{2}\right)^4 + \frac{157K^2-40K}{16}\left(\frac{p}{2}\right)^6 + \cdots \\
F_1 &= (K^2-K)\frac{p}{2} + \frac{11K^2-4K}{4}\left(\frac{p}{2}\right)^3 + \frac{5}{4}K^2\left(\frac{p}{2}\right)^4 + \frac{35K^2-15K}{8}\left(\frac{p}{2}\right)^5 + \frac{45K^2}{16}\left(\frac{p}{2}\right)^6 + \cdots \\
F_2 &= \frac{K-K^2}{2} - \frac{3K^2}{4}\left(\frac{p}{2}\right)^2 - \frac{6K^2-K}{2}\left(\frac{p}{2}\right)^4 + \cdots \\
F_3 &= \frac{2K-3K^2}{2}\frac{p}{2} + \frac{6K-15K^2}{4}\left(\frac{p}{2}\right)^3 + \frac{27K-53K^2}{8}\left(\frac{p}{2}\right)^5 + \cdots \\
F_4 &= \frac{K^2}{8} + \frac{K-3K^2}{2}\left(\frac{p}{2}\right)^2 + (K-3K^2)\left(\frac{p}{2}\right)^4 + 19\left(\frac{K-3K^2}{8}\right)\left(\frac{p}{2}\right)^6 + \cdots \\
F_5 &= \frac{K^3 p}{4} + \frac{3K^2-2K}{4}\left(\frac{p}{2}\right)^3 + \frac{49K^2-30K}{16}\left(\frac{p}{2}\right)^5 + \cdots \\
F_6 &= \frac{3K^2}{4}\left(\frac{p}{2}\right)^2 + \frac{6K^2-K}{2}\left(\frac{p}{2}\right)^4 + \frac{137K^2-26K}{16}\left(\frac{p}{2}\right)^6 + \cdots
\end{aligned}
\tag{31}
$$

In using these coefficients many of the terms containing higher powers of p may be discarded depending upon the number of significant figures desired by the computer. Table I indicates the agreement between ω^2/ω_0^2 calculated by the exact value given by equation (4) and the Fourier series approximation given in equations (24) and (31).

TABLE I

EXACT SOLUTION FOR ω^2/ω_o^2 VERSUS FOURIER APPROXIMATION

DATA:

$p = .25$

$K = .12$

Column A $\omega^2/\omega_o^2 = 1/[1 + K\sin^2\theta(1 + p\cos\theta/\cos\phi)^2]$.

Column B $\omega^2/\omega_o^2 = F_0 + F_1\cos\theta + F_2\cos 2\theta + F_3\cos 3\theta + \cdots$.

θ	A	B	Error
0	1.000000	1.000000	0
15	.987752	.987753	+ .000001
30	.957376	.957463	+ .000087
45	.922944	.923489	+ .000545
60	.897245	.898596	+.001351
75	.887004	.888842	+.001838
90	.892875	.894403	+.001546
105	.911137	.911978	+ .000841
120	.935953	.936254	+.000301
135	.961184	.961241	+.000057
150	.981994	.981995	+ .000001
165	.995402	.995399	−.000003
180	1.000000	1.000003	+.000003

CHAPTER VI

CONCLUSIONS

The assumption of constant crank angular velocity in a reciprocating engine has long been used in the dynamic analysis of that mechanism. Table I indicates that such an assumption is not true for the given machine taken in this analysis; nor, is it true for any machine. In addition this paper has developed a general expression (equations 24 and 31) for the angular velocity of an idealized engine. For vibration studies, harmonic analysis of forces and torques is necessary, which is the reason for developing a Fourier approximation rather than keeping the exact value expression given in (4).

It must be mentioned here that this work is deceiving in its brevity. However, many, many computations are necessary in determining the significance of the various constants and their combinations as they effect the magnitude of the coefficients that obtain in (31). If one more significant figure had been desired in (31) the work would have been more than doubled; serving merely to illustrate the law of diminishing returns. In any case, greater accuracy was not deemed necessary when the maximum error shown in Table I is less than 0.21% as compared to 11.3% for ω^2 assumed constant.

Inasmuch as the convergence of the final coefficients was impossible to prevision, several other factors influencing angular velocity were not considered in this solution. One such factor of decided significance is variation in driving force. Other factors requiring investigation are the effect of friction and varying load torque. Finally, the effect upon the error in Table I of varying K or p should be studied.

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APPENDIX I

THE KINEMATICS OF A SIMPLE SINGLE CYLINDER RECIPROCATING ENGINE 64

The motion of the simple reciprocating mechanism consists of, first, a rotating crank; and, second, a connecting rod one end of which moves in pure translation with the piston and the other end of which rotates with the crank (Fig. 2). Our primary interest here is the motion of the piston pin E which can be expressed in terms of the motion of the crank and the dimensions of the linkages. To analyze the motion of E, we proceed as follows:

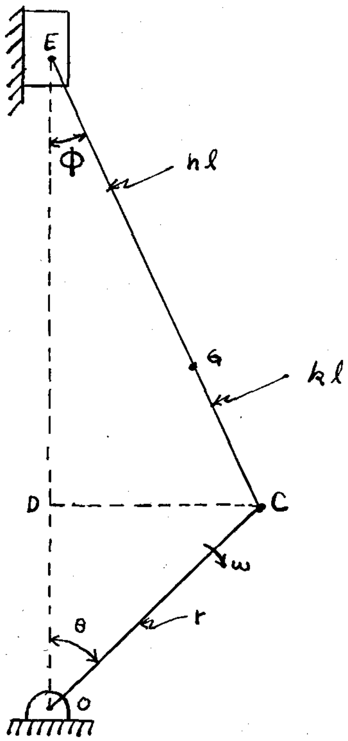

Fig. 2

$$CD = r \sin\theta = l \sin\phi. \qquad 7$$

$$\sin\phi = \frac{r}{l}\sin\theta = p\sin\theta. \qquad 32$$

where p is the ratio of crank r to connecting rod l. pl = r. Differentiating (32) with respect to time gives

$$\cos\phi \; d\phi/dt = p\cos\theta \; d\theta/dt,$$

or

$$d\phi/dt = (p\cos\theta/\cos\phi)\; d\theta/dt. \qquad 33$$

The displacement of the piston measured from its top dead center

position is

$$S_E = r + \ell - r\cos\theta - \ell\cos\phi. \quad 34$$

Differentiating (34) with respect to time results in

$$dS_E/dt = V_E = r\sin\theta \; d\theta/dt + \ell\sin\phi \; d\phi/dt. \quad 35$$

Substituting for $l\sin\phi$ from (7) and $d\phi/dt$ from (33) gives us

$$V_E = r\sin\theta \; d\theta/dt + (r\sin\theta \; \rho\cos\theta/\cos\phi) \; d\theta/dt. \quad 36$$

$$V_E = r\sin\theta \; (1 + \rho\cos\theta/\cos\phi) \; d\theta/dt, \quad 37$$

and since the angular velocity of the crank is $\omega = d\theta/dt$,

$$V_E = r\sin\theta(1 + \rho\cos\theta/\cos\phi)\,\omega. \quad 38$$

The angle ϕ can be completely eliminated from this notation by

$$\cos\phi = (1 - \sin^2\phi)^{1/2} = (1 - \rho^2\sin^2\theta)^{1/2}. \quad 8$$

Therefore,

$$V_E = r\sin\theta\,[1 + \rho\cos\theta/(1 - \rho^2\sin^2\theta)^{1/2}]\,\omega. \quad 39$$

From (35) and (7)

$$V_E = r\sin\theta \; (d\theta/dt + d\phi/dt), \quad 40$$

This can be differentiated with respect to time to give us

$$dV_\epsilon/dt = A_\epsilon = r\cos\theta\,\frac{d\theta}{dt}\left(\frac{d\theta}{dt}+\frac{d\phi}{dt}\right) + r\sin\theta\left(\frac{d^2\theta}{dt^2}+\frac{d^2\phi}{dt^2}\right). \qquad 41$$

If we square (33) we get

$$\left(\frac{d\phi}{dt}\right)^2 = \left(\rho^2\cos^2\theta/\cos^2\phi\right)\left(\frac{d\theta}{dt}\right)^2. \qquad 42$$

Also, differentiating (33) results in

$$-\sin\phi\left(\frac{d\phi}{dt}\right)^2 + \cos\phi\,\frac{d^2\phi}{dt^2} = -\rho\sin\theta\left(\frac{d\theta}{dt}\right)^2 + \rho\cos\theta\,\frac{d^2\theta}{dt^2}. \qquad 43$$

Simplifying this last expression and substituting (32) and (42) gives and expression for $d^2\phi/dt^2$,

$$\cos\phi\,\frac{d^2\phi}{dt^2} = \rho\cos\theta\,\frac{d^2\theta}{dt^2} - \rho\sin\theta\left(\frac{d\theta}{dt}\right)^2 + \sin\phi\left(\frac{d\phi}{dt}\right)^2.$$

$$\frac{d^2\phi}{dt^2} = \frac{\rho\cos\theta}{\cos\phi}\,\frac{d^2\theta}{dt^2} - \rho\frac{\sin\theta}{\cos\phi}\left(\frac{d\theta}{dt}\right)^2 + \frac{\rho\sin\theta}{\cos\phi}\left(\frac{\rho^2\cos^2\theta}{\cos^2\phi}\right)\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d^2\phi}{dt^2} = \rho\frac{\cos\theta}{\cos\phi}\,\frac{d^2\theta}{dt^2} + \left(\rho^3\frac{\sin\theta\cos^2\theta}{\cos^3\phi} - \rho\frac{\sin\theta}{\cos\phi}\right)\left(\frac{d\theta}{dt}\right)^2. \qquad 44$$

Substituting (33) and (44) into (41), we have

$$A_\epsilon = r\cos\theta\,\frac{d\theta}{dt}\left(\frac{d\theta}{dt} + \rho\,\frac{\cos\theta}{\cos\phi}\,\frac{d\theta}{dt}\right) + r\sin\theta\left[\frac{d^2\theta}{dt^2} + \frac{\rho\cos\theta}{\cos\phi}\,\frac{d^2\theta}{dt^2} + \left(\rho^3\frac{\sin\theta\cos^2\theta}{\cos^3\phi} - \frac{\rho\sin\theta}{\cos\phi}\right)\left(\frac{d\theta}{dt}\right)^2\right]$$

$$A_\epsilon = \left(\frac{d\theta}{dt}\right)^2\left(r\cos\theta + r\rho\frac{\cos^2\theta}{\cos\phi}\right) + \frac{d^2\theta}{dt^2}\left(r\sin\theta + r\rho\,\frac{\sin\theta\cos\theta}{\cos\phi}\right) + \left(\frac{d\theta}{dt}\right)^2\left(r\rho^3\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi} - r\rho\frac{\sin^2\theta}{\cos\phi}\right)$$

$$A_\epsilon = \left(r\rho^3\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi} - r\rho\frac{\sin^2\theta}{\cos\phi} + r\rho\frac{\cos^2\theta}{\cos\phi} + r\cos\theta\right)\left(\frac{d\theta}{dt}\right)^2 + \left(r\sin\theta + r\rho\frac{\sin\theta\cos\theta}{\cos\phi}\right)\frac{d^2\theta}{dt^2}. \qquad 45$$

Thus in (38) and (45) are expressed V_ϵ and A_ϵ completely in terms of r, 1, θ, and ω, recalling of course that (8) gives $\cos\phi$ as a function of θ, and that $d\theta/dt = \omega$, which in (37) and (45) are left unsubstituted.